Circle and Ellipse Fitting

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1 Introduction

The goal of this short document is to show how ellipse and circle fitting methods can be fomed by linear algorithm. Two basic problems are considered here, the circle and the ellipse fittings.

1.1 Circle fitting

The implicit formula of a circle is

$$(x - x_0)^2 + (y - y_0)^2 = R^2$$

where the circle center is $[x_0 \quad y_0]^T$ and the radius is denoted by R. This equation can be written as

$$x^2 - 2x_0x + x_0^2 + y^2 - 2y_0y + y_0^2 - R^2 = 0$$

If we have several point $[x_i \quad y_i]^T$, $i \in \{1, \ldots, N\}$ in the curve of the circle, the formula for all the points can be written as

$$\begin{bmatrix} 1 & -2x_1 & -2y_1 \\ 1 & -2x_2 & -2y_2 \\ \vdots & \vdots & \vdots \\ 1 & -2x_N & -2y_N \end{bmatrix} \begin{bmatrix} x_0^2 + y_0^2 - R^2 \\ x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} -x_1^2 - y_1^2 \\ -x_2^2 - y_2^2 \\ \vdots \\ -x_N^2 - y_N^2 \end{bmatrix}$$

Therefore it an imhomogeneous linear problem, the solution can be computed via the inverse, or pseudo-inverse in the over-determined case, of the left matrix.

1.1.1 Sphere fitting

The formula is very similar to circle fitting as it is its 3D 'version':

$$(x - x_0)^2 + (y - y_0)^2 + (z - y_0)^2 = R^2$$

if the spere center is $[x_0 \quad y_0 \quad z_0]^T$. Then considering 3D points $[x_i \quad y_i \quad z_i]^T i \in \{1, \ldots, N\}$ on the sphere surface, the following equation can be written:

$$x^2 - 2x_0x + x_0^2 + y^2 - 2y_0y + y_0^2 + z^2 - 2z_0z + z_0^2 - R^2 = 0.$$

The matrix-vector formula for the problem is

$$\begin{bmatrix} 1 & -2x_1 & -2y_1 & -2z_1 \\ 1 & -2x_2 & -2y_2 & -2z_2 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & -2x_N & -2y_N & -2z_N \end{bmatrix} \begin{bmatrix} x_0^2 + y_0^2 + z_0^2 - R^2 \\ x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} -x_1^2 - y_1^2 - z_1^2 \\ -x_2^2 - y_2^2 - z_2^2 \\ \vdots \\ -x_N^2 - y_N^2 - z_N^2 \end{bmatrix}$$

The solution can be obtained by the application of the pseudo-inverse.

1.2 Ellipse fitting

The general implicit formula for an ellipse is as follows:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

that is a quadratic curve. Parabolas, circles, hyperbolas are also quadratic, this formula represents an ellipse if $B^2 - 4AC < 0$.

The implicit firmula can be rewritten in a homogeneous linear for several points:

$$\begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_N^2 & x_Ny_N & y_N^2 & x_N & y_N & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \end{bmatrix} = 0$$

This formula can be written as

$$Mz = 0,$$

where

$$M = \begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_N^2 & x_Ny_N & y_N^2 & x_N & y_N & 1 \end{bmatrix}, z = \begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \end{bmatrix}.$$

This constraint $B^2 - 4AC < 0$ can also be written using vector and matrices. As the scale of the parameters is arbitrary, we can fix it as $B^2 - 4AC = -1$

$$\begin{bmatrix} A & B & C & D & E & F \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \end{bmatrix} = -1$$

Or, it compact form:

$$z^T K z = -1,$$

if

$$K = \begin{bmatrix} 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The ellipse fitting is a minimization problem $z = \arg_z \min \|Mz\|$ subject to $z^T K z = -1$. Therefore, Lagrangian technique can be applied:

$$z = \arg_{z,\lambda} \min \left\{ z^T M^T M z + \lambda \left(z^T K z + 1 \right) \right\}.$$

The gradient of the inner function is as follows:

$$2M^T M z + 2\lambda K z = 0.$$

Thus,

$$M^T M z = -\lambda K z.$$

This is a so-called generalized eigenvalue problem. At most six candidate solutions for z are obtained, the optimal one minimizes $z^T M^T M z$. As z has unit length, the length ambiguity can be removed by the constraint $z^T K z = -1$.

2 Curve normals

The curve normals are given by the gradients. The normal of a circle

$$J = (x - x_0)^2 + (y - y_0)^2 - R^2.$$

The normal is then

$$\nabla J = \begin{bmatrix} \frac{\partial J}{\partial x} \\ \frac{\partial J}{\partial y} \end{bmatrix} = \begin{bmatrix} 2(x - x_0) \\ 2(y - y_0) \end{bmatrix}.$$

If an ellipse is considered, then

$$J = Ax^2 + Bxy + Cy^2 + Dx + Ey + F.$$

Then the curve normal is as follows

$$\nabla J = \begin{bmatrix} \frac{\partial J}{\partial x} \\ \frac{\partial J}{\partial y} \end{bmatrix} = \begin{bmatrix} 2Ax + By + D \\ 2Cx + Bx + E \end{bmatrix}.$$