RQ-decomposition

November 2, 2020

Task: An arbitrary 3×3 matrix $\boldsymbol{A}$ is given:

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The task is to factorize it to the product of an upper triangular $\boldsymbol{K}$ and an orthonormal matrix $\mathbf{R}$.

For this purpose, the original matrix $\boldsymbol{A}$ is multiplied by three individual rotation matrix from the right. The first rotation, represented by matrix $\boldsymbol{R}_x$, is rotated around axis . The goal is to rotate matrix $\boldsymbol{A}$ as to modify the elment in the third row and second column to be zero:

$$\boldsymbol{A}\boldsymbol{R}_{X=} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & 0 & b_{33} \end{bmatrix} = \boldsymbol{B}$$

It involves the following equation:

$$a_{32}\cos\alpha + a_{33}\sin\alpha = 0$$

Therefore,

$$\tan\alpha = -\frac{a_{32}}{a_{33}}.$$

(If $a_{32} = 0$, then $\alpha = \frac{\pi}{2}$.)

After this, the result is rotated around axis Y in order to modify the element in the third row, first column to be zero:

$$\boldsymbol{B}\boldsymbol{R}_Y = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & 0 & b_{33} \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ 0 & 0 & c_{33} \end{bmatrix} = \boldsymbol{C}$$

It is possible if

$$b_{31}\cos\beta - b_{33}\sin\beta = 0.$$

(If $b_{33} = 0$, then $\beta = \frac{\pi}{2}$.)
Then

$$\tan\beta = \frac{b_{31}}{b_{33}}.$$

Finally the results has to be rotated around the third axis as

$$\boldsymbol{C}\boldsymbol{R}_Z = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ 0 & 0 & c_{33} \end{bmatrix} \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ 0 & d_{22} & d_{23} \\ 0 & 0 & d_{33} \end{bmatrix} = \boldsymbol{D}.$$

The element in the second row and first column becomes zero if

$$c_{21}\cos\gamma + c_{22}\sin\gamma = 0.$$

Then

$$\tan\gamma = -\frac{c_{21}}{c_{22}}.$$

(If $c_{32} = 0$, then $\gamma = \frac{\pi}{2}$.)
Matrix $\boldsymbol{D}$ is an upper triangular one. One can write

$$\boldsymbol{A}\boldsymbol{R}_x\boldsymbol{R}_y\boldsymbol{R}_z = \boldsymbol{D}.$$

$\boldsymbol{R}_x\boldsymbol{R}_y\boldsymbol{R}_z = \boldsymbol{R}$ is an orthonormal matrix. The decomposition is obtained by multiplying the matrix equation by the inverse of $\boldsymbol{R}$:

$$\boldsymbol{A} = \boldsymbol{D}\boldsymbol{R}^T.$$