Optimal Point Registration

December 10, 2019

1 Task

Given two point sets, both consist of N points. Tha aim is to find the similarity transformation between the images if the points p_i and q_i correspond to each other. The similarity transformation is represented by a rotation matrix $\boldsymbol{R}$, a translation vector $\boldsymbol{t}$ and a scale (scalar) parameter $s > 0$. The transformation estimation is to solve the following problem:

$$\arg_{s,\boldsymbol{R},\boldsymbol{t}} \min \sum_{i=1}^{N} \|s\boldsymbol{R}\boldsymbol{p}_i + \boldsymbol{t} - \boldsymbol{q}_i\|$$

subject to $s > 0$ and $\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}$.

2 Solution

The cost function is as follows

$$J = \sum \left(s\boldsymbol{R}\boldsymbol{p}_i + \boldsymbol{t} - \boldsymbol{q}_i\right)^T \left(s\boldsymbol{R}\boldsymbol{p}_i + \boldsymbol{t} - \boldsymbol{q}_i\right) =$$

$$s^2 \sum \boldsymbol{p}_i^T \boldsymbol{p}_i + N\boldsymbol{t}^T\boldsymbol{t} + \sum \boldsymbol{q_i^T}\boldsymbol{q_i} + 2s\boldsymbol{t}^T \sum \boldsymbol{R}\boldsymbol{p_i} - 2\boldsymbol{s} \sum \boldsymbol{q_i^T}\boldsymbol{R}\boldsymbol{p_i} - 2\boldsymbol{t}^T \sum \boldsymbol{q}_i$$

2.1 Optimal translation

Optimal translation can be obtained by taking the first derivative of cost function J with respect to translation vector $\boldsymbol{t}$.

$$\frac{\partial J}{\partial \boldsymbol{t}} = 2N\boldsymbol{t} + 2s \sum \boldsymbol{R}\boldsymbol{p}_i - 2 \sum \boldsymbol{q}_i = 0$$

Thus,

$$\boldsymbol{t} = \frac{\sum \boldsymbol{q}_i - \sum \boldsymbol{R}\boldsymbol{p}_i}{N} = \frac{\sum \boldsymbol{q}_i}{N} - \boldsymbol{R}\frac{\sum \boldsymbol{p}_i}{N}$$

If the coordinate systems are selected as to be in the gravity centers, then both $\frac{\sum \boldsymbol{q}_i}{N} = 0$ and $\frac{\sum \boldsymbol{p}_i}{N} = 0$. Thus, the optimal offset is zero. As the rotation $\boldsymbol{R}$ does not influence the center of gravity $\frac{\sum \boldsymbol{q}_i}{N}$, the optimal translation is the difference between the centers of gravity of the two point sets.

2.2 Optimal rotation

Let us eliminate the translation from the equations by selecting the origin as the centers of the gravity for both point clouds. Then the modified problem is as follows:

$$J = \sum \left(s\boldsymbol{R}\boldsymbol{p}_i - \boldsymbol{q}_i\right)^T \left(s\boldsymbol{R}\boldsymbol{p}_i - \boldsymbol{q}_i\right) = s^2 \sum \boldsymbol{p}_i^T \boldsymbol{p}_i + \sum \boldsymbol{q}_i^T \boldsymbol{q}_i - 2s \sum \boldsymbol{q}_i^T \boldsymbol{R}\boldsymbol{p}_i$$

As $s > 0$, the minimum of the cost function J w.r.t. rotation $\boldsymbol{R}$ is equivalent to maximize the term $\sum \boldsymbol{q}_i^T \boldsymbol{R}\boldsymbol{p}_i$.

$$\sum \boldsymbol{q}_i^T \boldsymbol{R}\boldsymbol{p}_i = tr(\boldsymbol{Q}\boldsymbol{R}\boldsymbol{P})$$

where

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{q_1^T} \\ \boldsymbol{q}_2^T \\ \vdots \\ \boldsymbol{q}_N^T \end{bmatrix}, \boldsymbol{P} = \begin{bmatrix} \boldsymbol{p}_1 & \boldsymbol{p}_2 & \dots & \boldsymbol{p}_N \end{bmatrix}$$

$$tr(\boldsymbol{Q}\boldsymbol{R}\boldsymbol{P}) = tr(\boldsymbol{R}\boldsymbol{P}\boldsymbol{Q})$$

The next step is the application of Singular Value Decomposition for the product matrix $\boldsymbol{P}\boldsymbol{Q}$:

$$\boldsymbol{P}\boldsymbol{Q} = \boldsymbol{U}\boldsymbol{S}\boldsymbol{V}^T$$

Then

$$tr(\boldsymbol{R}\boldsymbol{P}\boldsymbol{Q}) = tr(\boldsymbol{R}\boldsymbol{U}\boldsymbol{S}\boldsymbol{V}^{\boldsymbol{T}})$$

As $\boldsymbol{R}\boldsymbol{U}$ is a rotation (orthonormal) matrix as well as $\boldsymbol{V}^T$, and $\boldsymbol{S}$ is a diagonal one:

$$\boldsymbol{S} = \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix}$$

Let us use the following notations:

$$\boldsymbol{R}\boldsymbol{U} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \end{bmatrix}, \boldsymbol{V} = \begin{bmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \boldsymbol{b}_3 \end{bmatrix}$$

Then

$$tr(\boldsymbol{RUSV^T}) = tr(\boldsymbol{SV^T RU}) = \sum_{i=1}^{3} s_i \boldsymbol{a}_i^T \boldsymbol{b}_i$$

As $s_i > 0$ for all i, the maximization is equivalent to maximize $\sum \boldsymbol{a}_i^T \boldsymbol{b}_i$. $\boldsymbol{a}_i$ and $\boldsymbol{b}_i$ are row vectors of an orthonormal matrix, therefore the value of the dot product cannot exceed one. However, it can be reached if

$$\boldsymbol{RU} = \boldsymbol{V}.$$

This case maximizes the sum of the dot products. Therefore, the optimal solution is

$$\boldsymbol{R} = \boldsymbol{VU}^T.$$

2.3 Optimal scale

Finally, the problem is

$$J = \sum (s\boldsymbol{p}_i - \boldsymbol{q}_i)^T (s\boldsymbol{p}_i - \boldsymbol{q}_i) = s^2 \sum \boldsymbol{p}_i^T \boldsymbol{p}_i + \sum \boldsymbol{q}_i^T \boldsymbol{q}_i - 2s \sum \boldsymbol{q}_i^T \boldsymbol{p}_i$$

$$\frac{\partial J}{\partial s} = 2s \sum \boldsymbol{p}_i^T \boldsymbol{p}_i - 2 \sum \boldsymbol{p}_i^T \boldsymbol{q}_i = 0$$

Then

$$s = \frac{\sum \boldsymbol{p}_i^T \boldsymbol{q}_i}{\sum \boldsymbol{p}_i^T \boldsymbol{p}_i}$$

3 Remark

The method described above does not depend on the dimensionality of the problem, therefore it can be applied for all $(2, 3, \ldots, N)$ finite dimensions.