Point-ellipse distance

December 10, 2024

1 Problem Statement

Given an ellipse in the form

$$Au^2 + Bv^2 + Cuv + Du + Ev + F = 0.$$

This is equivalent to

$$\begin{bmatrix} u & v & 1 \end{bmatrix} M \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = 0,$$

where

$$M = \begin{bmatrix} A & C/2 & D/2 \\ C/2 & B & E/2 \\ D/2 & E/2 & F \end{bmatrix},$$

Or

$$M = \begin{bmatrix} \tilde{M} & a \\ a^T & F \end{bmatrix}$$

where $a = \begin{bmatrix} D/2 & E/2 \end{bmatrix}^T$ and

$$\tilde{M} = \begin{bmatrix} A & C/2 \\ C/2 & B \end{bmatrix}.$$

2 Closest point

The cost function to be minimized , , which yields the closest/further point(s) x w.r.t. x, is as follows:

$$J = (x - x_0)^T (x - x_0) + \lambda \left\{ x^T \tilde{M} x + 2a^T x + F \right\},$$

where

$$x = \begin{bmatrix} u \\ v \end{bmatrix}.$$

The gradient of the cost function is as follows:

$$\nabla_x J = 2\left(x - x_0\right) + 2\lambda \tilde{M} x + 2\lambda a = 0.$$

Then

$$\left(\lambda M \tilde{+} I\right) x = x_0 - \lambda a$$

x can be expressed as

$$x = \left(\lambda M \tilde{+} I\right)^{-1} \left(x_0 - \lambda a\right).$$

$$x = \begin{bmatrix} \lambda A + 1 & \lambda C/2 \\ \lambda C/2 & \lambda B + 1 \end{bmatrix}^{-1} \begin{bmatrix} u_0 - \lambda \frac{D}{2} \\ v_0 - \lambda \frac{E}{2} \end{bmatrix} \tag{1}$$

By applying the rule for the inversion of a 2×2 matrix, one can obtain

$$x = \frac{1}{\left(\lambda A + 1\right)\left(\lambda B + 1\right) - \lambda^2 C^2/4} \begin{bmatrix} \lambda B + 1 & -\lambda C/2 \\ -\lambda C/2 & \lambda A + 1 \end{bmatrix} \begin{bmatrix} u_0 - \lambda \frac{D}{2} \\ v_0 - \lambda \frac{E}{2} \end{bmatrix}.$$

It can be written in the form of

$$x = \frac{1}{P_3^2(\lambda)} \begin{bmatrix} P_1^2(\lambda) \\ P_2^2(\lambda) \end{bmatrix},$$

where

$$P_1^2(\lambda) = \begin{bmatrix} \lambda B + 1 & -\lambda C/2 \end{bmatrix} \begin{bmatrix} u_0 - \lambda \frac{D}{2} \\ v_0 - \lambda \frac{E}{2} \end{bmatrix} = \left(\lambda B + 1\right)\left(u_0 - \lambda \frac{D}{2}\right) - \lambda C/2 \left(v_0 - \lambda \frac{E}{2}\right) =$$

$$\left(EC/4 - DB/2\right)\lambda^2 + \left(Bu_0 - D/2 - Cv_0/2\right)\lambda + u_0$$

$$P_1^2(\lambda) = \begin{bmatrix} -\lambda C/2 & \lambda A + 1 \end{bmatrix} \begin{bmatrix} u_0 - \lambda \frac{D}{2} \\ v_0 - \lambda \frac{E}{2} \end{bmatrix} = -\lambda C/2 \left(u_0 - \lambda \frac{D}{2}\right) + \left(\lambda A + 1\right)\left(v_0 - \lambda \frac{E}{2}\right) =$$

$$\left(DC/4 - EA/2\right)\lambda^2 + \left(v_0 A - E/2 - u_0 C/2\right)\lambda + v_0$$

$$P_3^2(\lambda) = \left(\lambda A + 1\right)\left(\lambda B + 1\right) - \lambda^2 C^2/4 = \left(AB - C^2/4\right)\lambda^2 + \left(A + B\right)\lambda + 1$$

The constraint is then given in the following form:

$$x^T \tilde{M} x + 2a^T x + F = 0.$$

After substitution, the formula is

$$\left(\frac{1}{P_3^2(\lambda)}\right)^2 \left[\begin{array}{c} P_1^2(\lambda) \\ P_2^2(\lambda) \end{array}\right]^T \tilde{M} \left[\begin{array}{c} P_1^2(\lambda) \\ P_2^2(\lambda) \end{array}\right] + 2\left(\frac{1}{P_3^2(\lambda)}\right) a^T \left[\begin{array}{c} P_1^2(\lambda) \\ P_2^2(\lambda) \end{array}\right] + F = 0.$$

Moreover,

$$\left[\begin{array}{c} P_1^2(\lambda) \\ P_2^2(\lambda) \end{array}\right]^T \tilde{M} \left[\begin{array}{c} P_1^2(\lambda) \\ P_2^2(\lambda) \end{array}\right] + 2P_3^2(\lambda) a^T \left[\begin{array}{c} P_1^2(\lambda) \\ P_2^2(\lambda) \end{array}\right] + F\left(P_3^2(\lambda)\right)^2 = 0$$

This is a quartic polinomial. There are at most four roots for λ which should be evaluated. The values should be substituted to Eq.1, and the closest point is the final solution.