Lucas-Kanade tracker

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1 Introduction

The Lucas-Kanade tracker is a widely-used numerical template matching algorithm that is basically used to retrieve optical flow from video sequences. It optimized the parameters of an affine transformation between the two patches.

2 Problem Statement

There is two image, its pixels are denoted as $I_1(u_1, v_1)$ and $I_2(u_2, v_2)$. A patch is selected in the first image, the aim is to find the transformation that transform the original patch into a shape into a region on the second image, when the similarity is maximal. The location can be given via affine transformations:

$$p_2 = \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} (1+a_1)\,u_1 + a_3 v_1 + a_5 \\ a_2 u_1 + (1+a_4)\,v_1 + a_5 \end{bmatrix} = \begin{bmatrix} 1+a_1 & a_3 & a_5 \\ a_2 & 1+a_4 & a_6 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1+a_1 & a_3 & a_5 \\ a_2 & 1+a_4 & a_6 \end{bmatrix} \begin{bmatrix} p_1 \\ 1 \end{bmatrix} = A \begin{bmatrix} p_1 \\ 1 \end{bmatrix}$$

3 Algorithm

3.1 Taylor polynomial

We assume the first order Taylor polynomial , aka. linear approximation, of the image:

$$I_2(p_2 + \Delta p_2) = I_2(p_2) + \nabla^T I_2 \cdot \Delta p_2$$

where

$$\nabla I = \begin{bmatrix} \frac{\partial I}{\partial u} \\ \frac{\partial I}{\partial v} \end{bmatrix}.$$

$$\Delta p_2 = \begin{bmatrix} \Delta a_1 & \Delta a_3 & \Delta a_5 \\ \Delta a_2 & \Delta a_4 & \Delta a_6 \end{bmatrix} \begin{bmatrix} p_1 \\ 1 \end{bmatrix} = A \begin{bmatrix} p_1 \\ 1 \end{bmatrix}$$

Let us define the Jacobian matrix

$$\frac{\partial W}{\partial A} = \begin{bmatrix} \frac{\partial u_2}{\partial a_1} & \frac{\partial u_2}{\partial a_2} & \frac{\partial u_2}{\partial a_3} & \frac{\partial u_2}{\partial a_4} & \frac{\partial u_2}{\partial a_5} & \frac{\partial u_2}{\partial a_6} \\ \frac{\partial v_2}{\partial a_1} & \frac{\partial v_2}{\partial a_2} & \frac{\partial v_2}{\partial a_3} & \frac{\partial v_2}{\partial a_4} & \frac{\partial v_2}{\partial a_5} & \frac{\partial v_2}{\partial a_6} \end{bmatrix} =$$

$$\begin{bmatrix} u_2 & 0 & v_2 & 0 & 1 & 0 \\ 0 & u_2 & 0 & v_2 & 0 & 1 \end{bmatrix}$$

Then using the chain rule,

$$\Delta p_2 = \frac{\partial W}{\partial A} \Delta a$$

where

$$\Delta a = \begin{bmatrix} \Delta a_1 & \Delta a_2 & \Delta a_3 & \Delta a_4 & \Delta a_5 & \Delta a_6 \end{bmatrix}^T$$

Therefore,

$$I_2(p_2 + \Delta p_2) = I_2(p_2) + \nabla^T I_2(p_2) \frac{\partial W}{\partial A} \Delta a$$

3.2 Lucas-Kanade iteration

The aim of the step is to reduce the difference between the two patches. Formally, it can be written as a sum for all pixels inside the patch:

$$\arg_{a_1,\ldots,a_6} \min \sum \left(I_2\left(p_2 + \Delta p_2\right) - I_1\left(p_1\right)\right)^2$$

The term

$$I_2\left(p_2 + \Delta p_2\right) - I_1\left(p_1\right) = I_2\left(p_2\right) + \nabla^T I_2 \frac{\partial W}{\partial A} \Delta a - I_1\left(p_1\right)$$

The cost function can be written as

$$J = \sum \left(I_2\left(p_2 + \Delta p_2\right) - I_1\left(p_1\right)\right)^2$$

The optimal step is given as

$$\nabla_{\Delta p} J = 2 \sum \left[I_2\left(p_2\right) + \nabla^T I \frac{\partial W}{\partial A} \Delta a - I_1\left(p_1\right)\right] \nabla^T I \frac{\partial W}{\partial A} = 0$$

Let us introduce the vector

$$V^T = \nabla^T I \frac{\partial W}{\partial A} = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{bmatrix}$$

Then

$$\sum \left[I_2(p_2) - I_1(p_1) + V^T \Delta a \right] V = 0$$

Elementary modifications give the normal equation:

$$\sum \left[V^T \Delta a \right] V = \sum \left[I_1(p_1) - I_2(p_2) \right] V$$

It is a linear system for Δa :

$$\sum \left[V^T \Delta a \right] V = \sum V \left[V^T \Delta a \right] = \sum \begin{bmatrix} v_1 \\ v_{22} \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} \left(\begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{bmatrix} \begin{bmatrix} \Delta a_1 \\ \Delta a_2 \\ \Delta a_2 \\ \Delta a_4 \\ \Delta a_5 \\ \Delta a_6 \end{bmatrix} \right) =$$

$$\left(\sum V V^T \right) \Delta a$$

The normal equation is then

$$\sum \left[V^T \Delta a \right] V = \sum \left[I_1(p_1) - I_2(p_2) \right] V$$

$$\left(\sum V^T V \right) \Delta a = \sum \left[I_1(p_1) - I_2(p_2) \right] V$$

Therefore,

$$\Delta a = H^{-1} \sum \left[I_1(p_1) - I_2(p_2) \right] V$$

where

$$H = \left(\sum V^T V \right)$$

4 Condition numbers

The normal equation

$$H \Delta a = \sum \left[I_1(p_1) - I_2(p_2) \right] V$$

is well-conditioned, if the ratio of maximal and minimal eigenvalue of H is low. A special case, when the affine transformation is only limited to offset. In that case, $a_1 = a_2 = a_3 = a_4 = 0$, therefore

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a_5 \\ 0 & 1 & a_6 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ 1 \end{bmatrix} = \begin{bmatrix} u_1 + a_5 \\ u_2 + a_6 \end{bmatrix}$$

Then

$$\frac{\partial W}{\partial A} = \begin{bmatrix} \frac{\partial u_2}{\partial a_5} & \frac{\partial u_2}{\partial a_6} \\ \frac{\partial v_2}{\partial a_5} & \frac{\partial v_2}{\partial a_6} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

In that case,

$$V^T = \nabla^T I \frac{\partial W}{\partial A} = \begin{bmatrix} \frac{\partial I}{\partial u} \\ \frac{\partial I}{\partial v} \end{bmatrix}$$

For matrix H:

$$H = \left(\sum V^T V \right) = \sum \begin{bmatrix} \frac{\partial I}{\partial u} \\ \frac{\partial I}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{\partial I}{\partial u} & \frac{\partial I}{\partial v} \end{bmatrix} = \begin{bmatrix} \sum \left(\frac{\partial I}{\partial u}\right)^2 & \sum \left(\frac{\partial I}{\partial u}\right)\left(\frac{\partial I}{\partial v}\right) \\ \sum \left(\frac{\partial I}{\partial u}\right)\left(\frac{\partial I}{\partial v}\right) & \sum \left(\frac{\partial I}{\partial v}\right)^2 \end{bmatrix}$$

is obtained, that is called the local structure matrix. The problem is well-conditioned, if the smaller and larger eigenvalue of the local structure matrix are close to each other. Remark, that KLT corner detector