Lucas-Kanade tracker

June 6, 2025

1 Introduction

The Lucas-Kanade tracker is a widely-used numerical template matching algorithm that is basically used to retrieve optical flow from video sequences. It optimized the parameters of an affine transformation between the two patches.

2 Problem Statement

There is two image, its pixels are denoted as $I_1(u_1, v_1)$ and $I_2(u_2, v_2)$. A patch is selected in the first image, the aim is to find the transformation that transform the original patch into a shape into a region on the second image, when the similarity is maximal. The location can be given via affine transformations:

$$p_2 = \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} (1+a_1)\,u_1 + a_3 v_1 + a_5 \\ a_2 u_1 + (1+a_4)\,v_1 + a_5 \end{bmatrix} = \begin{bmatrix} 1+a_1 & a_3 & a_5 \\ a_2 & 1+a_4 & a_6 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1+a_1 & a_3 & a_5 \\ a_2 & 1+a_4 & a_6 \end{bmatrix} \begin{bmatrix} p_1 \\ 1 \end{bmatrix} = A \begin{bmatrix} p_1 \\ 1 \end{bmatrix}$$

3 Algorithm

3.1 Taylor polynomial

We assume the first order Taylor polynomial , aka. linear approximation, of the image:

$$I_2(p_2 + \Delta p_2) = I_2(p_2) + \nabla^T I_2 \cdot \Delta p_2$$

where

$$\nabla I = \begin{bmatrix} \frac{\partial I}{\partial u} \\ \frac{\partial I}{\partial v} \end{bmatrix}.$$

$$\Delta p_2 = \begin{bmatrix} \Delta a_1 & \Delta a_3 & \Delta a_5 \\ \Delta a_2 & \Delta a_4 & \Delta a_6 \end{bmatrix} \begin{bmatrix} p_1 \\ 1 \end{bmatrix} = A \begin{bmatrix} p_1 \\ 1 \end{bmatrix}$$

Let us define the Jacobian matrix

$$\frac{\partial W}{\partial A} = \begin{bmatrix} \frac{\partial u_2}{\partial a_1} & \frac{\partial u_2}{\partial a_2} & \frac{\partial u_2}{\partial a_3} & \frac{\partial u_2}{\partial a_4} & \frac{\partial u_2}{\partial a_5} & \frac{\partial u_2}{\partial a_6} \\ \frac{\partial v_2}{\partial a_1} & \frac{\partial v_2}{\partial a_2} & \frac{\partial v_2}{\partial a_3} & \frac{\partial v_2}{\partial a_4} & \frac{\partial v_2}{\partial a_5} & \frac{\partial v_2}{\partial a_6} \end{bmatrix} =$$

$$\begin{bmatrix} u_2 & 0 & v_2 & 0 & 1 & 0 \\ 0 & u_2 & 0 & v_2 & 0 & 1 \end{bmatrix}$$

Then using the chain rule,

$$\Delta p_2 = \frac{\partial W}{\partial A} \Delta a$$

where

$$\Delta a = \begin{bmatrix} \Delta a_1 & \Delta a_2 & \Delta a_3 & \Delta a_4 & \Delta a_5 & \Delta a_6 \end{bmatrix}^T$$

Therefore,

$$I_2(p_2 + \Delta p_2) = I_2(p_2) + \nabla^T I_2(p_2) \frac{\partial W}{\partial A} \Delta a$$

3.2 Lucas-Kanade iteration

The aim of the step is to reduce the difference between the two patches. Formally, it can be written as a sum for all pixels inside the patch:

$$\arg_{a_1,\ldots,a_6} \min \sum \left(I_2\left(p_2 + \Delta p_2\right) - I_1\left(p_1\right)\right)^2$$

The term

$$I_2\left(p_2 + \Delta p_2\right) - I_1\left(p_1\right) = I_2\left(p_2\right) + \nabla^T I_2 \frac{\partial W}{\partial A} \Delta a - I_1\left(p_1\right)$$

The cost function can be written as

$$J = \sum \left(I_2\left(p_2 + \Delta p_2\right) - I_1\left(p_1\right)\right)^2$$

The optimal step is given as

$$\nabla_{\Delta p} J = 2 \sum \left[I_2\left(p_2\right) + \nabla^T I \frac{\partial W}{\partial A} \Delta a - I_1\left(p_1\right)\right] \nabla^T I \frac{\partial W}{\partial A} = 0$$

Let us introduce the vector

$$V^T = \nabla^T I \frac{\partial W}{\partial A} = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{bmatrix}$$

Then

$$\sum \left[I_2 \left(p_2 \right) - I_1 \left(p_1 \right) + V^T \Delta a \right] V = 0$$

Elementary modifications give the normal equation:

$$\sum \left[V^T \Delta a \right] V = \sum \left[I_1 \left(p_1 \right) - I_2 \left(p_2 \right) \right] V$$

It is a linear system for Δa :

$$\sum \left[V^T \Delta a \right] V = \sum V \left[V^T \Delta a \right] = \sum \begin{bmatrix} v_1 \\ v_{22} \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} \left(\begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{bmatrix} \begin{bmatrix} \Delta a_1 \\ \Delta a_2 \\ \Delta a_2 \\ \Delta a_4 \\ \Delta a_5 \\ \Delta a_6 \end{bmatrix} \right) =$$

$$\left(\sum V V^T \right) \Delta a$$

The normal equation is then

$$\sum \left[V^T \Delta a \right] V = \sum \left[I_1 \left(p_1 \right) - I_2 \left(p_2 \right) \right] V$$

$$\left(\sum V^T V \right) \Delta a = \sum \left[I_1 \left(p_1 \right) - I_2 \left(p_2 \right) \right] V$$

Therefore,

$$\Delta a = H^{-1} \sum \left[I_1 \left(p_1 \right) - I_2 \left(p_2 \right) \right] V$$

where

$$H = \left(\sum V^T V \right)$$

4 Condition numbers

The normal equation

$$H \Delta a = \sum \left[I_1 \left(p_1 \right) - I_2 \left(p_2 \right) \right] V$$

is well-conditioned, if the ratio of maximal and minimal eigenvalue of H is low. A special case, when the affine transformation is only limited to offset. In that case, $a_1 = a_2 = a_3 = a_4 = 0$, therefore

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a_5 \\ 0 & 1 & a_6 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ 1 \end{bmatrix} = \begin{bmatrix} u_1 + a_5 \\ u_2 + a_6 \end{bmatrix}$$

Then

$$\frac{\partial W}{\partial A} = \begin{bmatrix} \frac{\partial u_2}{\partial a_5} & \frac{\partial u_2}{\partial a_6} \\ \frac{\partial v_2}{\partial a_5} & \frac{\partial v_2}{\partial a_6} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

In that case,

$$V^T = \nabla^T I \frac{\partial W}{\partial A} = \begin{bmatrix} \frac{\partial I}{\partial u} \\ \frac{\partial I}{\partial v} \end{bmatrix}$$

For matrix H:

$$H = \left(\sum V^T V \right) = \sum \begin{bmatrix} \frac{\partial I}{\partial u} \\ \frac{\partial I}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{\partial I}{\partial u} & \frac{\partial I}{\partial v} \end{bmatrix} = \begin{bmatrix} \sum \left(\frac{\partial I}{\partial u}\right)^2 & \sum \left(\frac{\partial I}{\partial u}\right)\left(\frac{\partial I}{\partial v}\right) \\ \sum \left(\frac{\partial I}{\partial u}\right)\left(\frac{\partial I}{\partial v}\right) & \sum \left(\frac{\partial I}{\partial v}\right)^2 \end{bmatrix}$$

is obtained, that is called the local structure matrix. The problem is well-conditioned, if the smaller and larger eigenvalue of the local structure matrix are close to each other. Remark, that KLT corner detector is very similar as it calculates the eigenvalues of the same matrix. The KLT condition is that the smallest eigenvalues should be large enough. As the eigenvalues are in linear ratio with the image contrast, KLT criterion is close to the algebraic one.