Cayley Parameterization of Rotations (SO(n))
A Detailed Explanation and Proof of Orthogonality

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1 Generalized Cayley Parameterization for SO(n)

The Cayley transform provides a rational parametrization for the rotation group SO(n) (the group of $n \times n$ orthogonal matrices with a determinant of $+1$). It is an algebraic alternative to exponential map representations.

1.1 The Cayley Transform Formula

Any rotation matrix $R \in \mathrm{SO}(n)$ that does **not** have -1 as an eigenvalue can be generated from an $n \times n$ skew-symmetric matrix $S \in \mathfrak{so}(n)$ using the Cayley transform:

$$R = (I + S)(I - S)^{-1}$$

where I is the $n \times n$ identity matrix.

1.2 The Cayley Parameters (Generalized Rodrigues Parameters)

The matrix S is skew-symmetric, defined by $S^T = -S$. This means the elements S_{ij} satisfy $S_{ii} = 0$ and $S_{ij} = -S_{ji}$. The number of independent real parameters required to define S is the number of elements above the main diagonal:

$$N_{\text{param}} = \frac{n(n-1)}{2}$$

These $\frac{n(n-1)}{2}$ values constitute the set of Cayley parameters necessary to specify a rotation in n dimensions. For example, in 3D ($n = 3$), $N_{\text{param}} = 3$, recovering the standard Rodrigues parameters.

2 Proof of Orthogonality: $R^T R = I$

To confirm that R is a rotation matrix, we must prove it is orthogonal, satisfying $R^T R = I$.

2.0.1 Step 1: Determine the Transpose R^T

We compute the transpose R^T. We use the properties of transposes and inverses: $(AB)^T = B^T A^T$, $(A^{-1})^T = (A^T)^{-1}$, and the skew-symmetry $S^T = -S$.

$$R^T = \left[(I + S)(I - S)^{-1}\right]^T$$

$$R^T = \left((I - S)^{-1}\right)^T (I + S)^T$$

$$R^T = \left((I - S)^T\right)^{-1} (I^T + S^T)$$

Substituting $I^T = I$ and $S^T = -S$:

$$R^T = \left(I - S^T\right)^{-1} (I - S)$$

$$R^T = (I - (-S))^{-1} (I - S)$$

$$\implies R^T = (I + S)^{-1}(I - S)$$

2.0.2 Step 2: Compute the Product $R^T R$

Substitute the derived expression for R^T and the original definition of R:

$$R^T R = \left[(I+S)^{-1}(I-S)\right]\left[(I+S)(I-S)^{-1}\right]$$

2.0.3 Step 3: Exploit Commutativity

Since I and S commute, the matrices $(I+S)$ and $(I-S)$ also commute:

$$(I+S)(I-S) = I - S^2 = (I-S)(I+S)$$

Since they commute, their inverses commute with each other and with the original matrices. This allows us to rearrange the terms:

$$R^T R = (I+S)^{-1}(I+S)\cdot(I-S)(I-S)^{-1}$$

Grouping the terms based on the definition of an inverse ($A^{-1}A = I$):

$$R^T R = \left[(I+S)^{-1}(I+S)\right]\left[(I-S)(I-S)^{-1}\right]$$

$$R^T R = I \cdot I$$

$$\implies R^T R = I$$

2.0.4 Conclusion

The result $R^T R = I$ proves that the matrix R generated by the Cayley parameters is an orthogonal matrix. Since the Cayley transform is a rational mapping that preserves the group identity, it is guaranteed that $\det(R) = +1$, confirming that $R \in \mathrm{SO}(n)$.

3 Cayley Parameterization for $\mathrm{SO}(2)$

For a 2-dimensional rotation, the rotation group is SO(2), which requires $N_{\text{param}} = \frac{2(2-1)}{2} = 1$ independent parameter. We denote this single Cayley parameter by s.

3.1 Defining the Skew-Symmetric Matrix S

The 2×2 skew-symmetric matrix $S \in \mathfrak{so}(2)$ is defined by the parameter s as:

$$S = \begin{pmatrix} 0 & -s \\ s & 0 \end{pmatrix}$$

3.2 The Cayley Transform

The rotation matrix R is derived using the Cayley transform:

$$R = (I+S)(I-S)^{-1}$$

where I is the 2×2 identity matrix.

3.2.1 Step 1: Compute $I+S$ and $I-S$

$$(I+S) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -s \\ s & 0 \end{pmatrix} = \begin{pmatrix} 1 & -s \\ s & 1 \end{pmatrix}$$

$$(I-S) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -s \\ s & 0 \end{pmatrix} = \begin{pmatrix} 1 & s \\ -s & 1 \end{pmatrix}$$

3.2.2 Step 2: Calculate the Inverse $(I-S)^{-1}$

The determinant of $(I-S)$ is $\det(I-S) = 1(1) - s(-s) = 1 + s^2$. The inverse is calculated using the adjoint matrix:

$$(I-S)^{-1} = \frac{1}{\det(I-S)}\mathrm{adj}(I-S) = \frac{1}{1+s^2}\begin{pmatrix} 1 & -s \\ s & 1 \end{pmatrix}$$

3.2.3 Step 3: Compute the Product R

We multiply $(I+S)$ by $(I-S)^{-1}$:

$$R = \begin{pmatrix} 1 & -s \\ s & 1 \end{pmatrix} \cdot \frac{1}{1+s^2} \begin{pmatrix} 1 & -s \\ s & 1 \end{pmatrix}$$

Pulling the scalar factor $\frac{1}{1+s^2}$ out and performing the matrix multiplication:

$$R = \frac{1}{1+s^2} \begin{pmatrix} (1)(1) + (-s)(s) & (1)(-s) + (-s)(1) \\ (s)(1) + (1)(s) & (s)(-s) + (1)(1) \end{pmatrix}$$

$$R = \frac{1}{1+s^2} \begin{pmatrix} 1-s^2 & -2s \\ 2s & 1-s^2 \end{pmatrix}$$

4 The Final 2×2 Rotation Matrix

The rotation matrix $R(s)$ parameterized by the Cayley parameter s is:

$$R(s) = \frac{1}{1+s^2} \begin{pmatrix} 1-s^2 & -2s \\ 2s & 1-s^2 \end{pmatrix}$$

4.1 Connection to the Rotation Angle θ

This matrix is equivalent to the standard trigonometric rotation matrix $R(\theta)$:

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

The relationship between the Cayley parameter s and the rotation angle θ is given by:

$$s = \tan\left(\frac{\theta}{2}\right)$$

The matrix $R(s)$ is obtained by substituting the half-angle trigonometric identities into $R(\theta)$.

4.2 Small proof for $s = \tan\left(\frac{\theta}{2}\right)$

For $R(\theta)$ and $R(s)$ to represent the same rotation, their corresponding elements must be equal.

4.2.1 Equating the Diagonal Elements $(\cos\theta)$

Comparing the $(1,1)$ element of $R(\theta)$ and $R(s)$:

$$\cos\theta = \frac{1-s^2}{1+s^2} \quad (*)$$

4.2.2 Equating the Off-Diagonal Elements $(\sin\theta)$

Comparing the $(2,1)$ element of $R(\theta)$ and $R(s)$:

$$\sin\theta = \frac{2s}{1+s^2} \quad (**)$$

The expressions $(*)$ and $(**)$ correspond directly to the standard trigonometric half-angle identities, which express the sine and cosine of an angle θ in terms of the tangent of the half-angle $\theta/2$.

The half-angle identities are:

$$\cos\theta = \frac{1-\tan^2(\theta/2)}{1+\tan^2(\theta/2)}$$

$$\sin\theta = \frac{2\tan(\theta/2)}{1+\tan^2(\theta/2)}$$

4.2.3 Comparison and Conclusion

By comparing the algebraic forms of equations ($*$) and ($**$) with the half-angle identities, we must conclude that the parameter s is identically equal to $\tan(\theta/2)$.

From the cosine relationship ($*$):

$$\frac{1-s^2}{1+s^2} = \frac{1-\tan^2(\theta/2)}{1+\tan^2(\theta/2)} \implies s^2 = \tan^2\left(\frac{\theta}{2}\right)$$

From the sine relationship ($**$):

$$\frac{2s}{1+s^2} = \frac{2\tan(\theta/2)}{1+\tan^2(\theta/2)} \implies s = \tan\left(\frac{\theta}{2}\right)$$

4.2.4 Result

Thus, the single Cayley parameter s is fundamentally related to the rotation angle θ by:

$$s = \tan\left(\frac{\theta}{2}\right)$$

This relationship is why the Cayley parameterization works: it uses a rational expression (the tangent of the half-angle) to compactly represent the trigonometric components of the rotation matrix.