Rodrigues' Rotation Formula

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December 11, 2025

Introduction

Rodrigues' rotation formula is an efficient algorithm used in 3D geometry and computer graphics to rotate a vector in space. It rotates a vector $\mathbf{v}$ by an angle θ around a fixed axis defined by a unit vector $\mathbf{k}$.

The Formula

The rotated vector $\mathbf{v}_{\mathrm{rot}}$ is given by:

$$\boxed{\mathbf{v}_{\mathrm{rot}} = \mathbf{v}\cos\theta + (\mathbf{k}\times\mathbf{v})\sin\theta + \mathbf{k}(\mathbf{k}\cdot\mathbf{v})(1-\cos\theta)} \tag{1}$$

Where:

- $\mathbf{v}$ is the original vector.
- $\mathbf{k}$ is the unit vector describing the rotation axis ($|\mathbf{k}| = 1$).
- θ is the angle of rotation (following the right-hand rule).

Geometric Derivation

The derivation relies on decomposing the vector $\mathbf{v}$ into two components relative to the rotation axis $\mathbf{k}$: one parallel and one perpendicular.

1. Vector Decomposition

We resolve $\mathbf{v}$ into a component parallel to $\mathbf{k}$ ($\mathbf{v}_{\|}$) and a component perpendicular to $\mathbf{k}$ ($\mathbf{v}_{\perp}$):

$$\mathbf{v} = \mathbf{v}_{\|} + \mathbf{v}_{\perp}$$

The parallel component is the projection of $\mathbf{v}$ onto $\mathbf{k}$:

$$\mathbf{v}_{\|} = (\mathbf{v}\cdot\mathbf{k})\mathbf{k} \tag{2}$$

The perpendicular component is simply the difference:

$$\mathbf{v}_{\perp} = \mathbf{v} - \mathbf{v}_{\|} = \mathbf{v} - (\mathbf{v}\cdot\mathbf{k})\mathbf{k} \tag{3}$$

2. Rotating the Components

When we perform the rotation:

- The parallel component $\mathbf{v}_{\|}$ lies on the axis of rotation, so it **remains unchanged**.
- The perpendicular component $\mathbf{v}_{\perp}$ rotates in the plane perpendicular to $\mathbf{k}$.

To describe the rotation of $\mathbf{v}_\perp$ in 2D, we need an orthogonal basis in the rotation plane. We already have $\mathbf{v}_\perp$. We can construct a second vector $\mathbf{w}$ that is perpendicular to both $\mathbf{k}$ and $\mathbf{v}_\perp$ using the cross product:

$$\mathbf{w} = \mathbf{k} \times \mathbf{v}_\perp = \mathbf{k} \times \mathbf{v}$$

(Note: $\mathbf{w}$ has the same magnitude as $\mathbf{v}_\perp$ but is rotated by 90°).

The rotated perpendicular component $\mathbf{v}_{\perp,\text{rot}}$ is a linear combination of $\mathbf{v}_\perp$ and $\mathbf{w}$:

$$\mathbf{v}_{\perp,\text{rot}} = \mathbf{v}_\perp \cos\theta + \mathbf{w} \sin\theta \tag{4}$$

3. Reassembling the Vector

The final rotated vector is the sum of the unchanged parallel part and the rotated perpendicular part:

$$\mathbf{v}_{\text{rot}} = \mathbf{v}_\parallel + \mathbf{v}_{\perp,\text{rot}}$$

Substituting the terms derived above:

$$\mathbf{v}_{\text{rot}} = \mathbf{v}_\parallel + (\mathbf{v}_\perp \cos\theta + (\mathbf{k} \times \mathbf{v}) \sin\theta)$$

Now, substitute $\mathbf{v}_\perp = \mathbf{v} - \mathbf{v}_\parallel$:

$$\mathbf{v}_{\text{rot}} = \mathbf{v}_\parallel + (\mathbf{v} - \mathbf{v}_\parallel) \cos\theta + (\mathbf{k} \times \mathbf{v}) \sin\theta$$

Group the terms by $\mathbf{v}_\parallel$:

$$\mathbf{v}_{\text{rot}} = \mathbf{v} \cos\theta + (\mathbf{k} \times \mathbf{v}) \sin\theta + \mathbf{v}_\parallel (1 - \cos\theta)$$

Finally, substitute $\mathbf{v}_\parallel = (\mathbf{v} \cdot \mathbf{k})\mathbf{k}$ to obtain the final formula:

$$\mathbf{v}_{\text{rot}} = \mathbf{v} \cos\theta + (\mathbf{k} \times \mathbf{v}) \sin\theta + \mathbf{k}(\mathbf{k} \cdot \mathbf{v})(1 - \cos\theta)$$

3. Matrix Representation

To find the rotation matrix R such that $\mathbf{v}' = R\mathbf{v}$, we convert the vector operations into matrix form.

1. **Identity:** $\mathbf{v} \to I\mathbf{v}$
2. **Cross Product:** $\mathbf{k} \times \mathbf{v} \to [\mathbf{k}]_\times \mathbf{v}$
   (Where $[\mathbf{k}]_\times$ is the skew-symmetric matrix of $\mathbf{k}$.)
3. **Tensor Product:** $\mathbf{k}(\mathbf{k} \cdot \mathbf{v}) \to (\mathbf{k}\mathbf{k}^T)\mathbf{v}$

The equation becomes:

$$R = I \cos(\theta) + [\mathbf{k}]_\times \sin(\theta) + (\mathbf{k}\mathbf{k}^T)(1 - \cos(\theta))$$

Final Compact Form

Using the identity $\mathbf{k}\mathbf{k}^T = I + [\mathbf{k}]_\times^2$, we arrive at the standard **Rodrigues' Rotation Formula**:

$$R = I + (\sin\theta)[\mathbf{k}]_\times + (1 - \cos\theta)[\mathbf{k}]_\times^2$$

Where $[\mathbf{k}]_\times$ is defined as:

$$[\mathbf{k}]_\times = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix}$$